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Theory and Methodology

A slacks-based measure of efficiency in data envelopment analysis

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Abstract

In this paper, we will propose a slacks-based measure (SBM) of efficiency in Data Envelopment Analysis (DEA). This scalar measure deals directly with the input excesses and the output shortfalls of the decision making unit (DMU) concerned. It is units invariant and monotone decreasing with respect to input excess and output shortfall. Furthermore, this measure is determined only by consulting the reference-set of the DMU and is not affected by statistics over the whole data set. The new measure has a close connection with other measures proposed so far, e.g., Charnes–Cooper–Rhodes (CCR), Banker–Charnes–Cooper (BCC) and the Russell measure of efficiency. The dual side of this model can be interpreted as profit maximization, in contrast to the ratio maximization of the CCR model. Numerical experiments show its validity as an efficiency measurement tool and its compatibility with other measures of efficiency.

Keywords: DEA; Efficiency; Slacks; Profit; Units invariant; Russell measure

1. Introduction

Since the innovative work by Charnes et al. (1978), studies in Data Envelopment Analysis (DEA) have been extensive: more than 1000 papers by 1996. One of the main objectives of DEA is to measure the efficiency of a Decision Making Unit (DMU) by a scalar measure ranging between zero (the worst) and one (the best). This scalar value is measured through a linear programming model. Specifically, the Charnes–Cooper–Rhodes (CCR) model deals with the ratio of multiple inputs and outputs in an attempt to gauge the relative efficiency of the DMU concerned among all the DMUs. This fractional program is solved by transforming it into an equivalent linear program using the Charnes–Cooper transformation. The optimal objective value (θ^*) is called the *ratio* (or *radial*) *efficiency* of the DMU. The optimal solution reveals, at the same time, the existence, if any, of excesses in inputs and shortfalls in outputs (called *slacks*). A DMU with the full ratio efficiency, $\theta^* = 1$, and with no *slacks* in any optimal solution is called *CCR-efficient*. Otherwise, the DMU has a disadvantage against the DMUs in its reference-set. Therefore, in discussing total efficiency, it is important to observe both the ratio

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efficiency and the slacks. Some attempts have been made to unify θ^* and slacks into a scalar measure, see Tone (1993) and Pastor (1995) among others.

Meanwhile, Charnes et al. (1985) developed the additive model of DEA, which deals directly with input excesses and output shortfalls. This model has no scalar measure (ratio efficiency) per se. Although this model can discriminate between efficient and inefficient DMUs by the existence of slacks, it has no means of gauging the depth of inefficiency, similar to θ^* in the CCR model. In an attempt to define inefficiency based on the slacks, Russell (1985, 1988), Pastor (1996), Lovell and Pastor (1995), Torgersen et al. (1996), Cooper and Pastor (1997), Cooper and Tone (1997), Thrall (1997) and others have proposed several formulae for finding a scalar measure. The following properties are considered as important in designing the measures.

- (P1) *Units invariant*: The measure should be invariant with respect to the units of data.
- (P2) *Monotone*: The measure should be monotone decreasing in each slack in input and output.
- (P3) *Translation invariant*: The measure should be invariant under parallel translation of the coordinate system applied (Ali and Seiford, 1990; Pastor, 1996).

In this paper, we further emphasize the original property of DEA:

- (P4) *Reference-set dependent*: The measure should be determined only by consulting the reference-set of the DMU concerned.

Given that DEA employs piece-wise linear efficient frontiers which are spanned by efficient DMUs, and that an inefficient unit is 'inefficient' with respect to DMUs in its reference-set, the measure of efficiency should be determined by the reference-set dependent values, as in the CCR and BCC (Banker et al., 1984) models. It should not be influenced by the extreme values, e.g., the minimum and the maximum of the data set, or by statistics covering the whole data set.

The new measure proposed in this paper satisfies the properties (P1), (P2) and (P4).

The rest of the paper is organized as follows. Section 2 proposes a new measure of efficiency (SBM) based on input excesses and output shortfalls, along with the computational scheme for solving the fractional program that defines SBM. We will show that the SBM can be interpreted as a product of input and output inefficiencies. The relationship with the CCR model is described in Section 3. The dual side of the SBM model is presented in Section 4, where it is shown that the model maximizes the virtual profit instead of the virtual ratio of the CCR model. This enables us to introduce economic aspects into the SBM: we can embed information on costs and prices into it. Furthermore, by restricting the feasible region of the dual variables (virtual costs and prices), we can make interpretations which are close to reality, similar to those using the assurance region method of Thompson et al. (1986) for the radial type models, e.g. the CCR and the BCC. In Section 5, an illustrative example is exhibited which compares the SBM with other CCR-type models. In Section 6, we will relax the positivity assumption of the data set. Finally, in Section 7, comparisons between the SBM and the Russell graph measure of technical efficiency (Russell, 1988; Färe et al., 1978, 1985) will be discussed from technical and economic viewpoints.

2. A slacks-based measure of efficiency

The definition of a slacks-based measure (SBM) of efficiency will be given, along with its interpretation as a product of input and output inefficiencies.

2.1. Definition and computational scheme of SBM

We will deal with n DMUs with the input and output matrices $X = (x_{ij}) \in \mathbb{R}^{m\times n}$ and $Y = (y_{ij}) \in \mathbb{R}^{s\times n}$, respectively. We assume that the data set is positive, i.e. $X > 0$ and $Y > 0$. (This assumption will be relaxed in Section 6.)

The production possibility set P is defined as

$$P = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \geqslant X\boldsymbol{\lambda},\ \boldsymbol{y} \leqslant Y\boldsymbol{\lambda},\ \boldsymbol{\lambda} \geqslant \boldsymbol{0}\}, \tag{1}$$

where $\boldsymbol{\lambda}$ is a nonnegative vector in $\mathbb{R}^n$. (We can impose some constraints on $\boldsymbol{\lambda}$, such as $\sum_{j=1}^n \lambda_j = 1$

(the BCC model), if it is needed to modify the production possibility set.)

We consider an expression for describing a certain DMU ($\boldsymbol{x}_o$, $\boldsymbol{y}_o$) as

$$\boldsymbol{x}_o = X\boldsymbol{\lambda} + \boldsymbol{s}^-, \quad (2)$$

$$\boldsymbol{y}_o = Y\boldsymbol{\lambda} - \boldsymbol{s}^+, \quad (3)$$

with $\boldsymbol{\lambda} \geqslant \boldsymbol{0}$, $\boldsymbol{s}^- \geqslant \boldsymbol{0}$ and $\boldsymbol{s}^+ \geqslant \boldsymbol{0}$. The vectors $\boldsymbol{s}^- \in \mathbb{R}^m$ and $\boldsymbol{s}^+ \in \mathbb{R}^s$ indicate the *input excess* and *output shortfall* of this expression, respectively, and are called *slacks*. From the conditions $X > 0$ and $\boldsymbol{\lambda} \geqslant \boldsymbol{0}$, it holds

$$\boldsymbol{x}_o \geqslant \boldsymbol{s}^-. \quad (4)$$

Using $\boldsymbol{s}^-$ and $\boldsymbol{s}^+$, we define an index ρ as follows:

$$\rho = \frac{1 - (1/m)\sum_{i=1}^{m} s_i^-/x_{io}}{1 + (1/s)\sum_{r=1}^{s} s_r^+/y_{ro}}. \quad (5)$$

It can be verified that ρ satisfies the properties (P1) (units invariant) and (P2) (monotone). Furthermore, from (4), it holds

$$0 < \rho \leqslant 1. \quad (6)$$

In an effort to estimate the efficiency of $(\boldsymbol{x}_o, \boldsymbol{y}_o)$, we formulate the following fractional program in $\boldsymbol{\lambda}$, $\boldsymbol{s}^-$ and $\boldsymbol{s}^+$.

[SBM]

$$\begin{aligned} \text{minimize} \quad & \rho = \frac{1-(1/m)\sum_{i=1}^{m} s_i^-/x_{io}}{1+(1/s)\sum_{r=1}^{s} s_r^+/y_{ro}} \\ \text{subject to} \quad & \boldsymbol{x}_o = X\boldsymbol{\lambda} + \boldsymbol{s}^-, \\ & \boldsymbol{y}_o = Y\boldsymbol{\lambda} - \boldsymbol{s}^+, \\ & \boldsymbol{\lambda} \geqslant \boldsymbol{0}, \quad \boldsymbol{s}^- \geqslant \boldsymbol{0}, \quad \boldsymbol{s}^+ \geqslant \boldsymbol{0}. \end{aligned} \quad (7)$$

[SBM] can be transformed into a linear program using the Charnes–Cooper transformation in the similar way as the CCR model. (See Charnes and Cooper, 1962; Charnes et al., 1978.)

Let us multiply a scalar variable $t(>0)$ to both the denominator and the numerator of (7). This causes no change in ρ. We adjust t so that the denominator becomes 1. Then this term is moved to constraints. The objective is to minimize the numerator. Thus, we have

[SBMt]

$$\begin{aligned} \text{minimize} \quad & \tau = t - \frac{1}{m}\sum_{i=1}^{m} ts_i^-/x_{io} \\ \text{subject to} \quad & 1 = t + \frac{1}{s}\sum_{r=1}^{s} ts_r^+/y_{ro}, \\ & \boldsymbol{x}_o = X\boldsymbol{\lambda} + \boldsymbol{s}^-, \\ & \boldsymbol{y}_o = Y\boldsymbol{\lambda} - \boldsymbol{s}^+, \\ & \boldsymbol{\lambda} \geqslant \boldsymbol{0}, \quad \boldsymbol{s}^- \geqslant \boldsymbol{0}, \quad \boldsymbol{s}^+ \geqslant \boldsymbol{0}, \quad t > 0. \end{aligned} \quad (8)$$

The problem given above is a nonlinear programming problem since it contains the nonlinear term ts_r^+ $(r = 1, \ldots, s)$. However we can transform it into a linear program as follows. Let us define

$$\boldsymbol{S}^- = t\boldsymbol{s}^-, \quad \boldsymbol{S}^+ = t\boldsymbol{s}^+ \quad \text{and} \quad \boldsymbol{\Lambda} = t\boldsymbol{\lambda}.$$

Then, [SBMt] becomes the following linear program in t, $\boldsymbol{S}^-$, $\boldsymbol{S}^+$ and $\boldsymbol{\Lambda}$:

[LP]

$$\begin{aligned} \text{minimize} \quad & \tau = t - \frac{1}{m}\sum_{i=1}^{m} S_i^-/x_{io} \\ \text{subject to} \quad & 1 = t + \frac{1}{s}\sum_{r=1}^{s} S_r^+/y_{ro}, \\ & t\boldsymbol{x}_o = X\boldsymbol{\Lambda} + \boldsymbol{S}^-, \\ & t\boldsymbol{y}_o = Y\boldsymbol{\Lambda} - \boldsymbol{S}^+, \\ & \boldsymbol{\Lambda} \geqslant \boldsymbol{0}, \quad \boldsymbol{S}^- \geqslant \boldsymbol{0}, \quad \boldsymbol{S}^+ \geqslant \boldsymbol{0}, \quad t > 0. \end{aligned} \quad (9)$$

Let an optimal solution of [LP] be

$$(\tau^*, t^*, \boldsymbol{\Lambda}^*, \boldsymbol{S}^{-*}, \boldsymbol{S}^{+*}).$$

Then, we have an optimal solution of [SBM] as defined by

$$\begin{aligned} & \rho^* = \tau^*, \quad \boldsymbol{\lambda}^* = \boldsymbol{\Lambda}^*/t^*, \quad \boldsymbol{s}^{-*} = \boldsymbol{S}^{-*}/t^*, \\ & \boldsymbol{s}^{+*} = \boldsymbol{S}^{+*}/t^*. \end{aligned} \quad (10)$$

Based on this optimal solution, we determine a DMU as being *SBM-efficient* as follows:

Definition 1 (*SBM-efficient*). A DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is SBM-efficient if $\rho^* = 1$.

This condition is equivalent to $\boldsymbol{s}^{-*} = \boldsymbol{0}$ and $\boldsymbol{s}^{+*} = \boldsymbol{0}$, i.e., no input excesses and no output shortfalls in any optimal solution.

For an SBM inefficient DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$, we have the expression:

$$\boldsymbol{x}_o = X\boldsymbol{\lambda}^* + \boldsymbol{s}^{-*},$$

$$\boldsymbol{y}_o = Y\boldsymbol{\lambda}^* - \boldsymbol{s}^{+*}.$$

The DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ can be improved and become efficient by deleting the input excess and augmenting the output shortfall as follows:

$$\boldsymbol{x}_o \leftarrow \boldsymbol{x}_o - \boldsymbol{s}^{-*}, \tag{11}$$

$$\boldsymbol{y}_o \leftarrow \boldsymbol{y}_o + \boldsymbol{s}^{+*}. \tag{12}$$

This operation is called the SBM-*projection*.

Based on $\boldsymbol{\lambda}^*$, we define the reference-set to $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ as follows:

Definition 2 (*Reference-set*). The set of indices corresponding to positive λ_j^*s is called the reference-set to $(\boldsymbol{x}_o, \boldsymbol{y}_o)$.

In the occurrence of multiple optimal solutions, the reference-set is not unique. We can choose any one for our purpose.

The reference-set R_o is

$$R_o = \{j \mid \lambda_j^* > 0\} \quad (j \in \{1, \ldots, n\}). \tag{13}$$

Using R_o, we can express $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ by

$$\boldsymbol{x}_o = \sum_{j \in R_o} \boldsymbol{x}_j \lambda_j^* + \boldsymbol{s}^{-*}, \tag{14}$$

$$\boldsymbol{y}_o = \sum_{j \in R_o} \boldsymbol{y}_j \lambda_j^* - \boldsymbol{s}^{+*}. \tag{15}$$

Since the SBM ρ^* depends only on $\boldsymbol{s}^{-*}$ and $\boldsymbol{s}^{+*}$, i.e., the reference-set dependent values, ρ^* is not affected by values attributed to other DMUs not in the reference-set. In this sense, ρ^* proposed in this paper is different from other efficiency measures which incorporate statistics over the whole data set.

2.2. Interpretation of SBM as product of input and output inefficiencies

The formula for ρ in (5) can be transformed into

$$\rho = \left(\frac{1}{m} \sum_{i=1}^{m} \frac{x_{io} - s_i^-}{x_{io}} \right) \left(\frac{1}{s} \sum_{r=1}^{s} \frac{y_{ro} + s_r^+}{y_{ro}} \right)^{-1}.$$

In the first term on the right-hand side, the ratio $(x_{io} - s_i^-)/x_{io}$ evaluates the relative reduction rate of input i and hence the first term corresponds to the mean reduction rate of inputs or *input inefficiency*. Similarly, in the second term, the ratio $(y_{ro} + s_r^+)/y_{ro}$ evaluates the relative expansion rate of output r and $(1/s)\sum(y_{ro} + s_r^+)/y_{ro}$ is the mean expansion rate of outputs. Its inverse, the second term measures *output inefficiency*. Thus, SBM ρ can be interpreted as the product of input and output inefficiencies.

2.3. Imposing bounds on the slacks

The projection formulas (11) and (12) may result in a large reduction (enlargement) in inputs (outputs) of the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$, which may not be allowed in the actual situations. In order to avoid such a difficulty, we can impose bounds on the slacks $\boldsymbol{s}^-$ and $\boldsymbol{s}^+$, such as

$$\boldsymbol{s}^- \leqslant \boldsymbol{s}_o^{-B} \quad \text{and} \quad \boldsymbol{s}^+ \leqslant \boldsymbol{s}_o^{+B}, \tag{16}$$

where the vector $\boldsymbol{s}_o^{-B}$ ($\boldsymbol{s}_o^{+B}$) is the upper bound of input reduction (output enlargement) of the DMU and should be specified for each DMU. Restricting the feasible region of slacks in this manner will give the efficiency measure a score not less than the original SBM score ρ^*.

3. Relationship with the CCR model

In this section, we will demonstrate that the SBM ρ^* is not greater than the CCR efficiency measure (θ^*) and that a DMU is SBM-efficient if and only if it is CCR-efficient.

3.1. SBM and the CCR measure

The (input-oriented) CCR model can be formulated as follows:

[CCR]

$$\begin{aligned} &\text{minimize} \quad \theta \\ &\text{subject to} \quad \theta \boldsymbol{x}_o = X\boldsymbol{\mu} + \boldsymbol{t}^-, && (17) \\ &\qquad \boldsymbol{y}_o = Y\boldsymbol{\mu} - \boldsymbol{t}^+, && (18) \\ &\qquad \boldsymbol{\mu} \geqslant \boldsymbol{0}, \quad \boldsymbol{t}^- \geqslant \boldsymbol{0}, \quad \boldsymbol{t}^+ \geqslant \boldsymbol{0}. \end{aligned}$$

Definition 3 (*CCR-efficient*). A DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is CCR-efficient, if the optimal objective value θ^* is equal to one and the optimal slacks $\boldsymbol{t}^{-*}$ and $\boldsymbol{t}^{+*}$ are zero for every optimal solution of [CCR].

For an inefficient DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$, the CCR *projection* is defined as

$$\boldsymbol{x}_o \Leftarrow \theta^* \boldsymbol{x}_o - \boldsymbol{t}^{-*}, \tag{19}$$

$$\boldsymbol{y}_o \Leftarrow \boldsymbol{y}_o + \boldsymbol{t}^{+*}.$$

Let an optimal solution of [CCR] be $(\theta^*, \boldsymbol{\mu}^*, \boldsymbol{t}^{-*}, \boldsymbol{t}^{+*})$. From (17), it holds

$$\boldsymbol{x}_o = X\boldsymbol{\mu}^* + \boldsymbol{t}^{-*} + (1 - \theta^*)\boldsymbol{x}_o, \tag{20}$$

$$\boldsymbol{y}_o = Y\boldsymbol{\mu}^* - \boldsymbol{t}^{+*}. \tag{21}$$

Let us define

$$\boldsymbol{\lambda} = \boldsymbol{\mu}^*, \tag{22}$$

$$\boldsymbol{s}^- = \boldsymbol{t}^{-*} + (1 - \theta^*)\boldsymbol{x}_o, \tag{23}$$

$$\boldsymbol{s}^+ = \boldsymbol{t}^{+*}. \tag{24}$$

Then, $(\boldsymbol{\lambda}, \boldsymbol{s}^-, \boldsymbol{s}^+)$ is feasible for [SBM] and its objective value is

$$\rho = \frac{1 - (1/m)\{\sum_{i=1}^m t_i^{-*}/x_{io} + m(1 - \theta^*)\}}{1 + (1/s)\sum_{r=1}^s t_r^{+*}/y_{ro}} = \frac{\theta^* - (1/m)\sum_{i=1}^m t_i^{-*}/x_{io}}{1 + (1/s)\sum_{r=1}^s t_r^{+*}/y_{ro}}. \tag{25}$$

We have the following theorem.

Theorem 1. *The optimal SBM ρ^* is not greater than the optimal CCR θ^*.*

Proof.

$$\rho^* \leqslant \rho = \frac{\theta^* - (1/m)\sum_{i=1}^m t_i^{-*}/x_{io}}{1 + (1/s)\sum_{r=1}^s t_r^{+*}/y_{ro}} \leqslant \theta^* - \frac{1}{m}\sum_{i=1}^m t_i^{-*}/x_{io} \leqslant \theta^*. \qquad \square$$

Notice that the coefficient $1/(m\, x_{io})$ of the input excess s_i^- in ρ plays a crucial role in validating Theorem 1.

Conversely, for an optimal solution $(\rho^*, \boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$ of [SBM], let us transform the constraints as

$$\theta \boldsymbol{x}_o = X\boldsymbol{\lambda}^* + (\theta - 1)\boldsymbol{x}_o + \boldsymbol{s}^{-*}, \tag{26}$$

$$\boldsymbol{y}_o = Y\boldsymbol{\lambda}^* - \boldsymbol{s}^{+*}. \tag{27}$$

Further, we add the constraint

$$(\theta - 1)\boldsymbol{x}_o + \boldsymbol{s}^{-*} \geqslant \boldsymbol{0}. \tag{28}$$

Then, $(\theta, \boldsymbol{\mu} = \boldsymbol{\lambda}^*, \boldsymbol{t}^- = (\theta - 1)\boldsymbol{x}_o + \boldsymbol{s}^{-*}, \boldsymbol{t}^+ = \boldsymbol{s}^{+*})$ is feasible for [CCR].

3.2. *SBM-efficiency and CCR-efficiency*

The relationship between CCR-efficiency and SBM-efficiency is demonstrated by the following theorem.

Theorem 2. *A DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is CCR-efficient, if and only if it is SBM-efficient.*

Proof. Suppose that $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is CCR-inefficient. Then, we have either $\theta^* < 1$ or ($\theta^* = 1$ and $(\boldsymbol{t}^{-*}, \boldsymbol{t}^{+*}) \neq (\boldsymbol{0}, \ \boldsymbol{0})$). From (25), in both cases, we have $\rho < 1$ for a feasible solution of [SBM]. Hence, $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is SBM-inefficient.

On the other hand, suppose that $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is SBM-inefficient. Then, it holds $(\boldsymbol{s}^{-*}, \ \boldsymbol{s}^{+*}) \neq (\boldsymbol{0}, \boldsymbol{0})$. By the statements (26) and (27), $(\theta, \ \boldsymbol{\mu} = \boldsymbol{\lambda}^*, \boldsymbol{t}^- = (\theta - 1)\boldsymbol{x}_o + \boldsymbol{s}^{-*}, \boldsymbol{t}^+ = \boldsymbol{s}^{+*})$ is feasible for [CCR], provided $(\theta - 1)\boldsymbol{x}_o + \boldsymbol{s}^{-*} \geqslant \boldsymbol{0}$. There are two cases.

Case 1 ($\theta = 1$ then $(\boldsymbol{t}^- = \boldsymbol{s}^{-*}, \boldsymbol{t}^+ = \boldsymbol{s}^{+*}) \neq (\boldsymbol{0}, \ \boldsymbol{0})$). In this case, an optimal solution for [CCR] is CCR-inefficient.

Case 2 ($\theta < 1$). In this case, $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is CCR-inefficient.

Therefore, CCR-inefficiency is equivalent to SBM-inefficiency. Since the definitions of *efficient* and *inefficient* are mutually exclusive, we have proved the theorem. $\square$

4. Observations on the dual problem

One of the important characteristics of DEA is its dual side, as represented by the dual program of the original linear program. This links the efficiency evaluation with the economic interpretation.

4.1. The dual program of the SBM model as profit maximization

The dual program of the problem [LP] in Section 2 can be expressed as follows, with the dual variables $\xi \in \mathbb{R}$, $\boldsymbol{v} \in \mathbb{R}^m$ and $\boldsymbol{u} \in \mathbb{R}^s$:

[DP]

$$\text{maximize} \quad \xi \tag{29}$$

$$\text{subject to} \quad \xi + \boldsymbol{v}\boldsymbol{x}_o - \boldsymbol{u}\boldsymbol{y}_o = 1, \tag{30}$$

$$-\boldsymbol{v}X + \boldsymbol{u}Y \leqslant \boldsymbol{0}, \tag{31}$$

$$\boldsymbol{v} \geqslant \frac{1}{m}[1/\boldsymbol{x}_o], \tag{32}$$

$$\boldsymbol{u} \geqslant \frac{\xi}{s}[1/\boldsymbol{y}_o], \tag{33}$$

where the notation $[1/\boldsymbol{x}_o]$ designates the row vector $(1/x_{1o}, 1/x_{2o}, \ldots, 1/x_{mo})$.

By Eq. (30), we can eliminate ξ. Then, this problem is equivalent to the following:

[DP′]

$$\text{maximize} \quad \boldsymbol{u}\boldsymbol{y}_o - \boldsymbol{v}\boldsymbol{x}_o \tag{34}$$

$$\text{subject to} \quad \boldsymbol{u}Y - \boldsymbol{v}X \leqslant \boldsymbol{0}, \tag{35}$$

$$\boldsymbol{v} \geqslant \frac{1}{m}[1/\boldsymbol{x}_o], \tag{36}$$

$$\boldsymbol{u} \geqslant \frac{1 - \boldsymbol{v}\boldsymbol{x}_o + \boldsymbol{u}\boldsymbol{y}_o}{s}[1/\boldsymbol{y}_o]. \tag{37}$$

The dual variables $\boldsymbol{v} \in \mathbb{R}^m$ and $\boldsymbol{u} \in \mathbb{R}^s$ can be interpreted as the virtual costs and prices of input and output items, respectively. The dual problem aims to find the optimal virtual costs and prices for the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ so that the profit $\boldsymbol{u}\boldsymbol{y}_j - \boldsymbol{v}\boldsymbol{x}_j$ does not exceed zero for any DMU (including $(\boldsymbol{x}_o, \boldsymbol{y}_o)$), and maximizes the profit $\boldsymbol{u}\boldsymbol{y}_o - \boldsymbol{v}\boldsymbol{x}_o$ for the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ concerned. Apparently, the optimal profit is at best zero and hence $\xi^* = 1$ for the SBM efficient DMUs.

Constraints (32) and (33) restrict the feasible $\boldsymbol{v}$ and $\boldsymbol{u}$ to the positive orthant. Using this framework, we can incorporate other important developments related to the virtual dual variables into the SBM model, e.g. the assurance region methods (Thompson et al., 1986, 1997; Thompson and Thrall, 1994), and the cone-ratio models (Charnes et al., 1990; Tone, 1997), among others. These modifications will contribute to the enhancement of the potential application of the model substantially. We will introduce this subject in Section 4.3.

4.2. Comparisons of dual programs in CCR and SBM models

The dual program of the CCR model can be expressed as

[DCCR]

$$\text{maximize} \quad \boldsymbol{\eta}\boldsymbol{y}_o \tag{38}$$

$$\text{subject to} \quad \boldsymbol{\xi}\boldsymbol{x}_o = 1, \tag{39}$$

$$-\boldsymbol{\xi}X + \boldsymbol{\eta}Y \leqslant \boldsymbol{0}, \tag{40}$$

$$\boldsymbol{\xi} \geqslant \boldsymbol{0}, \quad \boldsymbol{\eta} \geqslant \boldsymbol{0}. \tag{41}$$

This program originates from the ratio form CCR model (Charnes et al., 1978) below:

$$\text{maximize} \quad \frac{\boldsymbol{\eta}\boldsymbol{y}_o}{\boldsymbol{\xi}\boldsymbol{x}_o} \tag{42}$$

$$\text{subject to} \quad \frac{\boldsymbol{\eta}\boldsymbol{y}_j}{\boldsymbol{\xi}\boldsymbol{x}_j} \leqslant 1 \quad (\forall j), \tag{43}$$

$$\boldsymbol{\xi} \geqslant \boldsymbol{0}, \quad \boldsymbol{\eta} \geqslant \boldsymbol{0}. \tag{44}$$

Thus, the CCR model tries to find the virtual costs $\boldsymbol{\xi}$ and prices $\boldsymbol{\eta}$ so that the ratio $\boldsymbol{\eta}\boldsymbol{y}_o/\boldsymbol{\xi}\boldsymbol{x}_o$ is maximized, subject to the ratio constraint $\boldsymbol{\eta}\boldsymbol{y}_j/\boldsymbol{\xi}\boldsymbol{x}_j \leqslant 1$ for every DMU j.

The SBM model proposed in this paper deals with the *profit* instead of the *ratio* in the CCR model.

4.3. Taking account of cost/price information

Some may question the rationale behind maximizing the objective function (5) to evaluate the efficiency of DMU$_o$. The SBM will project the DMU to a "furthest" point on the efficient frontiers in the sense that the objective function is to be minimized by finding the maximal slacks. The upper bounds of slacks introduced in Section 2.3 will go some way to correspond to this question. However, the dual problem given above will contribute to making the SBM understandable in the economic interpretation. Since the dual variables $\boldsymbol{v}$ and $\boldsymbol{u}$ can be interpreted as the virtual costs and prices of the corresponding inputs and outputs, we can embed cost/price information into the dual problem. A similar situation occurs in the assurance region model of Thompson et al. (1986). In many applications, information on costs and prices is unavailable, or is subject to change by chance. Hence it may be reasonable to set the lower and upper bounds of the ratio of virtual costs of two items as follows:

$$l_{ij} \leqslant \frac{v_j}{v_i} \leqslant u_{ij}. \tag{45}$$

Similarly, for outputs, we can impose the lower (upper) bound on the ratio of the virtual prices of outputs i to j as follows:

$$L_{ij} \leqslant \frac{u_j}{u_i} \leqslant U_{ij}. \tag{46}$$

Thus, we have an SBM with an assurance region of $(\boldsymbol{v}, \boldsymbol{u})$ as follows:

[DP″]

$$\begin{aligned} &\text{maximize} \quad \boldsymbol{u}\boldsymbol{y}_o - \boldsymbol{v}\boldsymbol{x}_o && (47)\\ &\text{subject to} \quad \boldsymbol{u}Y - \boldsymbol{v}X \leqslant \boldsymbol{0}, && (48)\\ &\qquad \boldsymbol{v} \geqslant \frac{1}{m}[1/\boldsymbol{x}_o], && (49)\\ &\qquad \boldsymbol{u} \geqslant \frac{1 - \boldsymbol{v}\boldsymbol{x}_o + \boldsymbol{u}\boldsymbol{y}_o}{s}[1/\boldsymbol{y}_o], && (50)\\ &\qquad \boldsymbol{v}P \leqslant \boldsymbol{0}, && (51)\\ &\qquad \boldsymbol{u}Q \leqslant \boldsymbol{0}, && (52) \end{aligned}$$

where

$$P = \begin{pmatrix} l_{12} & -u_{12} & l_{13} & -u_{13} & \cdots & \cdots & \cdots & \cdots \\ -1 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -1 & 1 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and

$$Q = \begin{pmatrix} L_{12} & -U_{12} & L_{13} & -U_{13} & \cdots & \cdots & \cdots & \cdots \\ -1 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -1 & 1 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

The matrices P and Q have dimensions $m \times n_1$ and $s \times n_2$, respectively, where n_1 and n_2 are determined by the number of assurance region constraints imposed.

The dual of the abovesaid program can be expressed, using $t \in \mathbb{R}$, $\boldsymbol{S}^- \in \mathbb{R}^m$, $\boldsymbol{S}^+ \in \mathbb{R}^s$, $\boldsymbol{\Lambda} \in \mathbb{R}^n$, $\boldsymbol{\Phi} \in \mathbb{R}^{n_1}$, $\boldsymbol{\Pi} \in \mathbb{R}^{n_2}$ as variables, as below.

[SBM-AR]

$$\begin{aligned} &\text{minimize} \quad \eta = t - \frac{1}{m}\sum_{i=1}^{m} S_i^-/x_{io} && (53)\\ &\text{subject to} \quad 1 = t + \frac{1}{s}\sum_{r=1}^{s} S_r^+/y_{ro},\\ &\qquad t\boldsymbol{x}_o = X\boldsymbol{\Lambda} + \boldsymbol{S}^- - P\boldsymbol{\Phi},\\ &\qquad t\boldsymbol{y}_o = Y\boldsymbol{\Lambda} - \boldsymbol{S}^+ + Q\boldsymbol{\Pi},\\ &\qquad \boldsymbol{\Lambda} \geqslant \boldsymbol{0}, \quad \boldsymbol{S}^- \geqslant \boldsymbol{0}, \quad \boldsymbol{S}^+ \geqslant \boldsymbol{0},\\ &\qquad t > 0, \quad \boldsymbol{\Phi} \geqslant \boldsymbol{0}, \quad \boldsymbol{\Pi} \geqslant \boldsymbol{0}. \end{aligned}$$

Let an optimal solution of [SBM-AR] be $(\eta^*, t^*, \boldsymbol{\Lambda}^*, \boldsymbol{S}^{-*}, \boldsymbol{S}^{+*}, \boldsymbol{\Phi}^*, \boldsymbol{\Pi}^*)$. Then we have an optimal solution of the SBM under the assurance region constraints (51) and (52) as defined by

$$\rho^* = \eta^*, \quad \boldsymbol{\lambda}^* = \boldsymbol{\Lambda}^*/t^*, \quad \boldsymbol{s}^{-*} = \boldsymbol{S}^{-*}/t^*,$$
$$\boldsymbol{s}^{+*} = \boldsymbol{S}^{+*}/t^*, \quad \boldsymbol{\phi}^* = \boldsymbol{\Phi}^*/t^*, \quad \boldsymbol{\pi}^* = \boldsymbol{\Pi}^*/t^*. \tag{54}$$

A DMU is called “SBM-AR efficient” if and only if $\rho^* = 1$. The SBM projection under the assurance region constraints can be attained by

$$\widehat{\boldsymbol{x}}_o \leftarrow \boldsymbol{x}_o - \boldsymbol{s}^{-*} - P\boldsymbol{\phi}^* (= X\boldsymbol{\lambda}^*), \tag{55}$$
$$\widehat{\boldsymbol{y}}_o \leftarrow \boldsymbol{y}_o + \boldsymbol{s}^{+*} + Q\boldsymbol{\pi}^* (= Y\boldsymbol{\lambda}^*). \tag{56}$$

5. An illustrative example

In this section, we compare the SBM with the CCR model using a simple illustrative problem

Table 1
Two inputs and two outputs data

DMU	Input1	Input2	Output1	Output2
A	4	3	2	3
B	6	3	2	3
C	8	1	6	2
D	8	1	6	1
E	2	4	1	4

consisting of five DMUs with two inputs and two outputs. The data set is exhibited in Table 1.

5.1. Comparisons with the CCR model

First, we compare the results obtained by applying the input-oriented CCR model with those using the SBM, as displayed in Table 2. The notations in the table correspond to those in (17), (18) and (7). Hence, $\boldsymbol{t}^-$ and $\boldsymbol{t}^+$ denote the optimal slacks for the CCR model, while $\boldsymbol{s}^-$ and $\boldsymbol{s}^+$ denote those for the SBM. Since the CCR score is a radial measure and takes no account of slacks, DMU D has the full efficiency score $\theta = 1$, although it has a shortfall of $t_2^+ = 1$ in Output2 against C. However, when this factor was taken into account, the SBM score of D came down to $\rho = 0.667$. Similarly, the SBM scores of A and B were worse than those in the CCR model due to the existence of slacks. Finally, CCR-efficient C and E remained at the efficient status under SBM evaluations, as claimed by Theorem 2.

The optimal weights of both models are exhibited in Table 3.

5.2. Comparisons with the CCR model under assurance region constraints

In order to compare the change in efficiency owing to the addition of assurance region constraints, we augmented the following ratio constraints to the pairs of inputs and outputs for the abovesaid sample problem:

$$0.5 \leqslant \frac{v_1}{v_2} \leqslant 1 \quad \text{and} \quad 0.5 \leqslant \frac{u_1}{u_2} \leqslant 1. \tag{57}$$

Table 2
Comparisons with the CCR model

DMU	CCR					SBM				
	Score	Slack				Score	Slack			
	θ	t_1^-	t_2^-	t_1^+	t_2^+	ρ	s_1^-	s_2^-	s_1^+	s_2^+
A	0.9	0	0	0.4	0	0.798	0	0.357	0.714	0
B	0.833	0	0	1.5	0	0.568	0	0.643	2.286	0
C	1	0	0	0	0	1	0	0	0	0
D	1	0	0	0	1	0.667	0	0	0	1
E	1	0	0	0	0	1	0	0	0	0

Table 3
Comparisons of optimal weights

DMU	CCR					SBM				
	Score	Weight				Score	Weight			
	θ	ξ_1	ξ_2	η_1	η_2	ρ	v_1	v_2	u_1	u_2
A	0.9	0.04	0.28	0	0.3	0.798	0.181	0.167	0.199	0.207
B	0.833	0.037	0.260	0	0.278	0.568	0.135	0.167	0.142	0.199
C	1	0.104	0.167	0.111	0.167	1	0.063	0.5	0.083	0.25
D	1	0.125	0	0.167	0	0.667	0.062	0.5	0.056	0.333
E	1	0.125	0.188	0.125	0.219	1	0.411	0.125	0.5	0.205

Similarly, we imposed the same weight constraints to $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ for the CCR model in (38)–(41) as follows:

$$0.5 \leqslant \frac{\xi_1}{\xi_2} \leqslant 1 \quad \text{and} \quad 0.5 \leqslant \frac{\eta_1}{\eta_2} \leqslant 1. \tag{58}$$

Comparisons of the results of these problems under the assurance region constraints, i.e., CCR-AR and SBM-AR, are exhibited in Table 4. In the CCR model, the complementary slackness conditions between slacks and weights assert that

$$\xi_i^* t_i^{-*} = 0 \ (\forall i) \quad \text{and} \quad \eta_r^* t_r^{+*} = 0 \ (\forall r).$$

Hence, it is observed that DMUs A and B have $\eta_1^* = 0$ and D has $\eta_2^* = 0$. This means that Output1 was not accounted for efficiency evaluations of A and B. On the other hand, the SBM has the following complementarity conditions at optimality:

$$\left(v_i^* - \frac{1}{m\, x_{io}}\right) s_i^{-*} = 0 \quad (\forall i)$$

and

$$\left(u_r^* - \frac{\rho^*}{s\, y_{io}}\right) s_r^{+*} = 0 \quad (\forall r),$$

where ρ^* is the SBM score. Thus, v_i^* and u_r^* are bounded below by $1/(m\, x_{io})$ and $\rho^*/(s\, y_{ro})$, respectively, and are all positive. Furthermore, the weighted input satisfies the relation $v_i^* x_{io} \geqslant 1/m$ $(\forall i)$ and the weighted output satisfies $u_r^* y_{ro} \geqslant \rho^*/s$ $(\forall r)$. It can be concluded that in the SBM all inputs and outputs contribute to evaluation of efficiency, at least at the levels mentioned above.

It is observed that in the CCR-AR, A, B and D showed reduced scores. This was caused by the addition of the assurance region constraints. C and E remained at the status of full efficiency. On the other hand, in the SBM-AR, even E dropped from the status of efficiency by dint of these assurance region conditions. This was also caused by the difference of the models, i.e., the ratio maximization and the profit maximization.

6. How to deal with zeros in data

So far, we have assumed that the data set is positive, i.e., $X > 0$ and $Y > 0$. In this section, we relax this assumption and show how to deal with zeros in the input/output data and even negative output data. This will considerably expand the applicability of the SBM to real world problems, which essentially involve systematic zeros in the input/output data matrix.

6.1. Zeros in input data

If $\boldsymbol{x}_o$ has zero elements, we can neglect the slacks corresponding to these zeros. Suppose, for example, that $x_{1o} = 0$. Then, the first constraint leads to:

$$\sum_{j=1}^{n} x_{1j}\lambda_j + s_1^- = x_{1o} = 0.$$

Table 4
Comparisons with the CCR model under assurance region constraints

DMU	CCR-AR					SBM-AR				
	Score	Weight				Score	Weight			
	θ	ξ_1	ξ_2	η_1	η_2	ρ	v_1	v_2	u_1	u_2
A	0.857	0.111	0.186	0.107	0.214	0.667	0.167	0.333	0.167	0.333
B	0.702	0.091	0.152	0.088	0.175	0.5	0.125	0.25	0.125	0.25
C	1	0.111	0.111	0.125	0.125	1	0.25	0.5	0.25	0.5
D	0.875	0.111	0.111	0.125	0.125	0.5	0.25	0.5	0.25	0.5
E	1	0.167	0.167	0.182	0.205	0.667	0.333	0.667	0.333	0.667

Hence, we have $s_1^- = 0$ for every feasible solution. Thus, we can delete s_1^- from the set of variables to be determined by the model. Correspondingly, in the objective function, the term s_1^-/x_{1o} is removed and m should be reduced by 1 ($m \to m - 1$). Notice that the abovesaid constraint should be kept in the set of constraints.

6.2. Zeros in output data

Suppose that $\boldsymbol{y}_o$ has $y_{1o} = 0$. Then, the first output-constraint leads to

$$\sum_{j=1}^{n} y_{1j}\lambda_j - s_1^+ = y_{1o} = 0.$$

There are two important cases to be considered:

Case 1. The target DMU possesses no function to produce the first output. In this case, we can delete the term s_1^+/y_{1o} from the objective function, since s_1^+ has no role in evaluating the efficiency of the DMU. The number of terms (s) in the objective function should be reduced by 1 ($s \to s - 1$).

Case 2. The target DMU has a function with the potential of producing the first output but does not ulilize it. In this case, we may replace y_{1o} by a small positive number or by

$$y_{1o} \leftarrow \frac{1}{10}\min\{y_{1j} \mid y_{1j} > 0,\ j = 1,\ldots,n\}.$$

It should be remembered that the term s_1^+/y_{1o} in the objective function has the role of a penalty in this case, and that $1/y_{1o}$ should be sufficiently large.

Finally, negative output data can be dealt using the same approach adopted for handling zeros in output data.

7. Comparisons with the Russell measure of efficiency

The Russell graph measure of technical efficiency (Färe et al., 1985) is defined as follows:

[Russell]

$$\text{minimize} \quad \frac{1}{m+s}\left(\sum_{i=1}^{m}\theta_i + \sum_{r=1}^{s}\frac{1}{\phi_r}\right) \tag{59}$$

$$\begin{aligned} \text{subject to} \quad & \theta_i x_{io} \geqslant \sum_{i=1}^{m} x_{ij}\lambda_j \quad (i = 1,\ldots,m), \\ & \phi_r y_{ro} \leqslant \sum_{r=1}^{s} y_{rj}\lambda_j \quad (r = 1,\ldots,s), \\ & 0 \leqslant \theta_i \leqslant 1, \quad \phi \geqslant 1 \ \forall i, r, \\ & \lambda_j \geqslant 0 \quad (j = 1,\ldots,n). \end{aligned}$$

Let us change notations by introducing $s_{io}^-(\geqslant 0)$ $(i = 1,\ldots,m)$ and $s_{ro}^+(\geqslant 0)$ $(r = 1,\ldots,s)$, which satisfy the relations

$$\theta_i = \frac{x_{io} - s_{io}^-}{x_{io}} \ (\leqslant 1) \quad \text{and} \quad \phi_r = \frac{y_{ro} + s_{ro}^+}{y_{ro}} \ (\geqslant 1).$$

Then, after some mathematical manipulations, the abovesaid Russell formulation comes to

[Russell′]

$$\text{minimize} \quad \frac{1}{m+s}\left(m\left(1 - \frac{1}{m}\sum_{i=1}^{m} s_{io}^-/x_{io}\right) + \sum_{r=1}^{s}\frac{1}{1 + s_{ro}^+/y_{ro}}\right) \tag{60}$$

$$\begin{aligned} \text{subject to} \quad & \boldsymbol{x}_o = X\boldsymbol{\lambda} + \boldsymbol{s}_o^-, \\ & \boldsymbol{y}_o = Y\boldsymbol{\lambda} - \boldsymbol{s}_o^+, \\ & \boldsymbol{\lambda} \geqslant \boldsymbol{0}. \end{aligned}$$

The SBM measure is similar to Russell's in that both deal directly with slacks and give an efficiency measure between 0 and 1. Also both are monotone decreasing with respect to slacks. However, the SBM differs from the Russell measure in the following ways.

1. The Russell measure needs to be solved using a nonlinear programming problem, while the SBM can be solved using the linear program technology.
2. The Russell measure is an average of an arithmetic mean and a harmonic mean, and its meaning is unclear, while the SBM can be interpreted as explained in Section 2.2.
3. More importantly, the Russell measure has no definite dual program and it is difficult to attain

an economic interpretation. Hence it is also not fitted for embedding cost/price information into the model, whereas the SBM has a well-defined dual program which can be interpreted as a virtual profit maximization problem as described in Section 4.
4. Färe et al. (1978, 1994) introduced the input (output) oriented Russell measure, which is equivalent to the input (output) oriented SBM, if we deal with only the numerator (denominator) of the SBM in (7).

8. Conclusion

This article proposed a scalar measure SBM of efficiency in DEA. In contrast to the CCR and BCC measures, which are based on the proportional reduction (enlargement) of input (output) vectors and which do not take account of slacks, the SBM deals directly with input excess and output shortfall. Although the additive model has the (weighted) sum of slacks as its objective and can discriminate between *efficient* and *inefficient* DMUs, it has no means to gauge the depth of inefficiency per se. In this framework, the SBM shows a sharp contrast to CCR, BCC and, other measures proposed so far.

The SBM satisfies such properties as unit invariance and monotone with respect to slacks. Furthermore, it is reference-set dependent, i.e. the measure is determined only by its reference-set and is not affected by statistics over the whole data set. Also, this model can be modified to cope with input or output-orientation as special cases. These cases are the same as those used in the input (output) oriented Russell measure of technical efficiency. The dual program revealed that SBM tries to find the maximum virtual profit, unlike the CCR model, which attempts to find the maximum ratio of virtual output over virtual input.

The numerical example demonstrated the compatibility of SBM with other measures and its potential applicability for practical purposes.

Although this study concentrated on the basic characteristics of the proposed model, further theoretical research and applications should be developed in diverse areas, including studies in the combinations of this method with other recent developments in DEA.

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